# Newton's shell theorem

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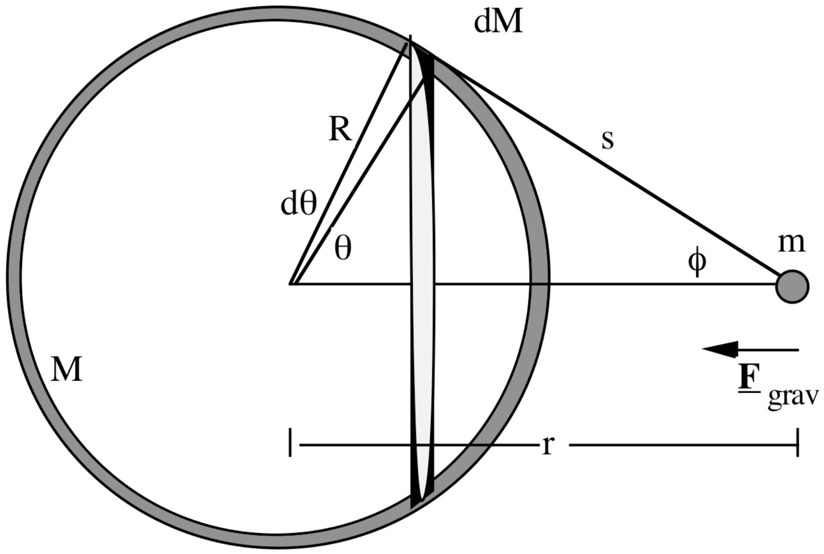
This theorem shows that a spherically symmetric body, mass $M_{body}$, exerts a gravitational force on an external object, mass m, that equals the force between m and a point mass $M_{body}$, located at the centre of the spherical distribution. Many astronomical objects (especially stars and planets) are nearly spherically symmetric. Consequently, we can calculate the gravitational forces between them using their masses and the distance between their centres. Let's see why.

Consider first a thin spherically symmetric shell (dark shading) of mass M and radius R and a point mass m at r from its centre, as shown. First we consider the case r > R. We need to integrate the gravitational attraction between m and all parts of M.

For the integration, we slice the shell up into narrow rings (light shading), so that all points in the ring are the same distance, s, from m. Let the ring subtend an angle dθ, as shown.




We define σ, the area density or mass per unit area of the shell: $\sigma = \frac{M}{4\pi R^2}$ . So the mass of the ring is dM is σ times its area. Its circumference is $2\pi R\sin\theta$ and its width is $Rd\theta$, so

$$dM = \sigma.2\pi R\sin\theta.Rd\theta = \frac{M\sin\theta.d\theta}{2}$$

The force exerted by each part of the ring is a vector, and we must add these vectors. From symmetry, however, the gravitational force on m will be along the axis of the ring, so we need only add the vectors on the axis. Each tiny part along the perimeter of the ring exerts a force towards it, and the axial component of each force includes a factor cosϕ.

$$dF = \frac{GmdM}{s^2}\cos\phi = \frac{GmM}{2s^2}\sin\theta\cos\phi\, d\theta$$

From the cosine rule, we can write

$$\cos\theta = \frac{r^2 + R^2 - s^2}{2rR} \quad \text{and} \quad \cos\phi = \frac{r^2 + s^2 - R^2}{2rs}$$

All three of θ, ϕ and s vary. Let's make s the independent variable. Differentiating both sides of the equation for cos θ gives

$$\sin\theta\, d\theta = \frac{s}{rR}\, ds$$

We can then subsitute this equation, plus the expression for cos ϕ in our expression for dF

$$dF = \frac{GmM}{2s^2}\frac{r^2 + s^2 - R^2}{2rs}\frac{s}{rR}\, ds = \frac{GmM}{4Rr^2}\frac{r^2 - R^2 + s^2}{s^2}\, ds$$

To include the whole shell, s varies from (r–R) to (r+R), so the force due to the whole shell is

$$F = \frac{GmM}{4Rr^2} \int_{s = r-R}^{r+R} \frac{r^2 - R^2 + s^2}{s^2} \, ds$$

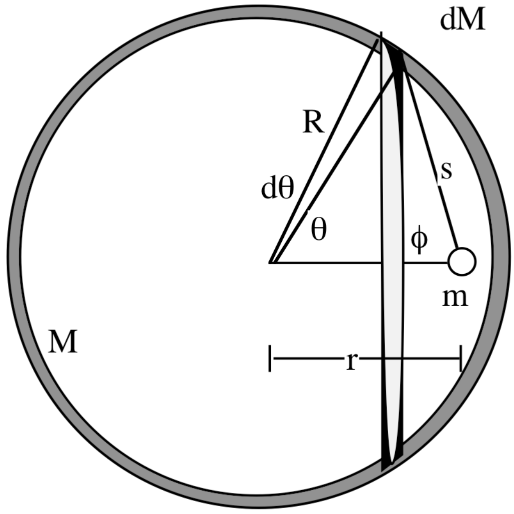
The integrand as the form $(\text{constant}^2 + s^2)/s^2$, so the integral is a standard one. The definite integral has the value 4R, so we obtain the simple expression

$$F = \frac{GmM}{r^2}$$

For a spherically symmetric distribution of mass, we can then integrate over R. Each shell gives a similar expression.

Now look at the case where $r < R$, i.e. when m is inside the hollow shell. In this case, integrating over the sphere requires that s go from R-r to R+r: this changes the sign of the lower limit of integration. In this case, the integral gives zero. So, inside a hollow shell, the total gravitational field due to the shell is zero.

This may seem odd: in this diagram, most of the shell is to the left of m. However, some of the parts to the right of m are closer to m, and these effects cancel out.




Combining the two results, we see that, at a point inside a symmetric distribution of mass, only the mass closer to the centre contributes to the gravitational field.